THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. IX. MAY, 1902. No. 5.

MODELS OF THE WEIERSTRASS SIGMA FUNCTION AND THE ELLIPTIC INTEGRAL OF THE SECOND KIND.

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In 1886, Professor Dyck, of the Polytechnicum of Munich, requested the construction of models to represent the elliptic functions. In accordance with this request H. Burkhardt and M. W. Wildbrett constructed seven models, two of which are for the case in which the invariant J of the binary quartic is equal to zero.

In 1899, Professor Klein, while lecturing on the automorphic functions, requested that the series be extended by making corresponding models of the sigma function, and of the elliptic integral of the second kind.

By Professor Klein's invitation I undertook the task which consists of three steps. These may be named as follows:

(1) Analytic representation;
(2) Numerical calculation;
(3) Mechanical construction.

(1) ANALYTIC REPRESENTATION. The sigma function can most easily be expressed in terms of theta functions, which in turn are defined by trigonometric series with exponential functions for coefficients. For values of the variable within the parallelogram containing the origin two terms of the series are sufficient for numerical values correct to three places of decimals. For values of the

variable not in the initial parallelogram more terms are necessary, but the form of the function can be easily expressed by a recurring formula. Since three terms are sufficient to obtain approximations correct to eight places of decimals in the first parallelogram, it was found that three terms are sufficient for all values included in nine parallelgrams about the origin.

The two components of the variable are chosen as rectangular coördinates in a plane, and the real part of the function is erected as ordinate from the point.

Thus, if $u+iv=f(x+iy)$, then $u=f_1(x, y)$, and $v=f_2(x, y)$.

The following theorems can now be easily established when the invariant J is put equal to zero.

The line $x=0$, $u=0$ lies on the surface $u=f_1(x, y)$.

The u-surface is symmetric as to the plane $x=0$.

Finally, $u(-x, y)=-u(x, y)$.

To obtain the v surface, we have the relation

$$f(iz)=if(z),$$

hence the v surface can be obtained from the u surface by rotating the latter through a positive right angle about the line $x=0$, $y=0$.

(2) Numerical Calculation. By means of logarithmic and trigonometric tables and also tables for exponential and hyperbolic functions the numerical calculation is a matter of routine. I gave y a constant value, then calculated the value of u for values of x taken at intervals of $\frac{1}{24}$ of the period; this was repeated for values of y taken at intervals of $\frac{1}{24}$ of the second period. The numerical values were thus determined over nine parallelograms. The niveau lines are now easily found by marking those points on the parallelogram which have the same value of u. By drawing the curves which connect points of the same value of u the entire system is determined. To one familiar with topographical drawings this system of niveau curves furnishes a very good representation of the function.

(3) Mechanical Construction. When $J=0$ the two periods are numercally equal and their ratio is i, hence they may be represented by the sides of a square. For length of period 6 c.m. was chosen so that the base is 18 x 18 c.m. The curves $y=c$, $x=k$ were drawn on cross-section paper ruled to millemeters. The squares of paper were pasted on sheets of zinc and the zinc was then sawed through along the curve. Only alternate plane sections were drawn, at intervals of 5 m.m. These curves in zinc were then dove-tailed together, carefully squared and firmly soldered. A base was then sawed out corresponding to $z=-9$ (c.m.) and fixed to the previous part. This frame was filled with plaster, which was then shaved off while still green until the curves $y=c$, $x=k$ were exposed. After the plaster became more firmly set the exact contour was given to the surface by checking off the ordinates of the intermediate points. The niveau lines were easily marked by adjusting a horizontal scratch-awl so as to make a slight furrow in the plaster at a constant distance above the base. In this way

niveau lines at intervals of 1 c.m. were designated. A rather more elaborate mechanism is necessary to mark the lines of flow v=constant. These lines are so drawn that their vertical projections on the x, y plane are orthogonal to the niveau curves u=constant.

The model, thus constructed, was sent to Hrn. Kreittmäyr in Munich, for duplication. During the summer of 1899 I worked for several days in his workshops but didn't learn enough to make a complete report of this interestiug process.

The elliptic integral of the second kind is defined as the logarithmic derivative of the sigma function. The procedure for this function was essentially the same as for the sigma function, but here four parallelograms of periods are sufficient, as the function is periodic with regard to one of the periods of the elliptic function. The three theorems for sigma still hold for this function.

One fact is at once apparent from the model of the sigma function; for values of z within the first parallelogram it is of practical value in numerical calculations, but comparatively cumbersome for the next parallelogram and utterly worthless for parallelograms still further removed.

Both models are published by Martin Schilling, in Halle, successor to S. Brill, in Darmstadt.

MERIDIAN AND TRANSVERSE SECTIONS OF HELICOIDS OF UNIFORM PITCH.

By ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Miss.

The following discussion was suggested by the treatment of this subject given in MacCord's Descriptive Geometry. In addition to establishing analytically the facts stated in that text-book, other properties of the curves of intersection have been determined.

The helicoid here considered is a surface generated by a straight line always tangent to a right circular cylinder, inclined at a constant angle to the axis of the cylinder, and having two simultaneous uniform motions—one of revolution about the axis, the other of translation parallel thereto. The locus of the point of tangency of this generatrix with the cylindrical surface is a common helix.

Let AB be the axis of the cylinder, C the center of a right section which intersects the helix OH at O, Q any point on OH, RT the corresponding position of the generatrix piercing the plane ABS, determined by the axis and O, at P, and meeting the plane of the right section at T.

Then P is any point of the meridian section of the helicoid and T any point of the transverse section.